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Correlation Inequalities for a Class of Even Ferromagnets

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We present rigorous correlation inequalities for connected n-point functions in a class of even ferromagnets. The class includes spin-1/2 Ising models and scalar field models with potential function V which is even and continuously differentiable with V' convex on $[0, \infty)$. These inequalities are obtained by pushing ahead with the method of Ellis, Monroe, and Newman at its maximum.

KEY WORDS: Correlation inequality; Ising model; scalar field models.

1. INTRODUCTION

The correlation inequality is a powerful tool in the rigorous study of statistical mechanical system (see Ref. 1 and references therein). Many correlation inequalities are known for the spin-1/2 classical Ising model. Some of them have been shown to hold for some other models[(2)] including lattice scalar field models. They have played important role in proofs of the triviality[(3)] or nontriviality[(4)] of the $(\varphi^4)_d$ field models for $d > 4$ or $d < 4$, respectively.

In view of these points, we think it very important to try to find new correlation inequalities maximally within the reach of the presently known methods. In this spirit, we have exploited the method of Refs. 5, 6, and 7 and extend it to find new correlation inequalities for the connected n-point function U_n.

We consider models of general even ferromagnets with pair interactions in a positive external field. The models are defined by a finite family

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of real-valued random variables $\Phi \equiv \{\varphi_i; i=1,..., N\}$, whose joint probability distribution $\mu_{h_1 \cdots h_N}$ on R^N has the form

$$d\mu_{h_1 \cdots h_N}(\Phi) = Z(\{h_i\})^{-1} \exp[-H(\Phi)] \prod_{i=1}^{N} d\nu_i(\varphi_i) \tag{1.1}$$

where $H(\Phi)$ is the Hamiltonian defined by

$$H(\Phi) = -\sum_{1 \leqslant i \leqslant j \leqslant N} J_{ij}\varphi_i\varphi_j - \sum_{1 \leqslant i \leqslant N} h_i\varphi_i \tag{1.2}$$

and $Z(\{h_i\}) \equiv Z(h_1,..., h_N)$ is the partition function defined by

$$Z(\{h_i\}) = \int_{R^N} \exp[-H(\Phi)] \prod_{i=1}^{N} d\nu_i(\varphi_i) \tag{1.3}$$

The indices i and j label sites in a lattice $\Lambda = \{1,..., N\}$ of N sites. φ_i denotes the spin of the ith site, $J_{ij} \geqslant 0$ the ferromagnetic interaction strength between φ_i and φ_j, and $h_i \geqslant 0$ the nonnegative external magnetic field strength at the ith site.

Let $\mathscr{E}$ be the set of all the even probability measures satisfying

$$\int \exp(b\varphi^2)\, d\nu(\varphi) < \infty \tag{1.4}$$

for some $b > 0$.

We define a subclass $\mathscr{G}$ of $\mathscr{E}$ as follows.

Definition. Given $\nu \in \mathscr{E}$, let $\Phi^{(a)}, a=1,..., 4$ be four independent copies of a random variable Φ distributed by ν, not μ in (1.1). Let $\mathbf{\Phi} = (\Phi^{(1)},..., \Phi^{(4)})$ and $\mathbf{m} = (m^{(1)},..., m^{(4)})$, where each $m^{(a)}$ is a nonnegative integer. We define an orthogonal matrix B by

$$B = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \tag{1.5}$$

and use the following notations:

$$(B\mathbf{\Phi})^{(a)} = \sum_{b=1}^{4} B_{ab}\Phi^{(b)}, \qquad (B\mathbf{\Phi})^{\mathbf{m}} = \prod_{a=1}^{4} \{(B\mathbf{\Phi})^{(a)}\}^{m^{(a)}} \tag{1.6}$$

Now we define the EMN (Ellis–Monroe–Newman) class $\mathscr{G}$ by

$$\mathscr{G} = \{ v \in \mathscr{E};\ E_0[(B\mathbf{\Phi})^{\mathbf{m}}] \geqslant 0 \text{ for all } \mathbf{m} \} \tag{1.7}$$

where the expectation $E_0[F]$ is defined by

$$E_0[F] = \int F(\Phi^{(1)},..., \Phi^{(4)}) \prod_{a=1}^{4} \prod_{i \in \Lambda} dv(\varphi^{(a)_i}) \tag{1.8}$$

For the explicit examples of the measure $dv \in \mathscr{G}$, refer to Refs. 6, 7, and 8.

In Section 2 the method of Ref. 6 is reviewed with our notations and generalizations.

In Section 3 we present new correlation inequalities on U_3, U_4, and U_6 and state some applications of them.

The results on U_5 and U_8 are rather complicated. Hence we collect them in tables at the end.

2. THE METHOD OF PROOFS

For each $i = 1,..., N$, let $\varphi_i^{(a)}$, $a = 1,..., 4$ be four independent copies of a family of random variables, $\{\varphi_1,..., \varphi_N\}$ distributed by the measure

$$d\rho(\Phi) = Z(\{0\})^{-1} \exp\left(\sum_{i,j \in \Lambda} J_{ij} \varphi_i \varphi_j \right) \prod_{i=1}^{N} dv_i(\varphi_i) \tag{2.1}$$

We denote the (tensor product) expectation $E[F]$, for a function $F(\Phi^{(1)},..., \Phi^{(4)})$, by

$$E[F] = \int d\rho(\Phi^{(1)}) \cdots d\rho(\Phi^{(4)})\, F(\Phi^{(1)},..., \Phi^{(4)}) \tag{2.2}$$

Hence

$$\int d\mu(\Phi^{(1)}) \cdots d\mu(\Phi^{(4)})\, F(\Phi^{(1)},..., \Phi^{(4)})$$

$$= [Z(\{0\})/Z(\{h_i\})]^4\, E\left[F(\Phi^{(1)},..., \Phi^{(4)}) \exp\left(\sum_i \mathbf{h}_i \cdot \boldsymbol{\varphi}_i \right) \right] \tag{2.3}$$

With $\mathbf{h}_i \equiv (h_i,..., h_i)$, define

$$\mathbf{s}_i \equiv (s^{(1)_i},..., s^{(4)_i}) = \mathbf{h}_i B^T = (2h_i, 0, 0, 0) \tag{2.4}$$

Then, expanding the exponential, we have

$$E\left\{(B\Phi)^{\mathbf{P}} \exp\left[\sum_{i=1}^{N} \mathbf{s}_i \cdot (B\boldsymbol{\varphi}_i)\right]\right\}$$

$$= E\left\{\prod_{a=1}^{4} [(B\Phi)^{(a)}]^{P(a)} \exp\left[2 \sum_{i=1}^{N} h_i (B\varphi_i)^{(1)}\right]\right\}$$

$$\equiv \sum_{\substack{m_i^{(1)}=0 \\ i=1,\dots,N}}^{\infty} \prod_{i=1}^{N} \frac{(2h_i)^{m_i(1)}}{m_i(1)!} E\left\{\prod_{a=1}^{4} [(B\Phi)^{(a)}]^{P(a)} \prod_{j=1}^{N} [(B\varphi_j)^{(1)}]^{m_j(1)}\right\}$$

$$= \sum_{\substack{m_i^{(1)}=0 \\ i=1,\dots,N}}^{\infty} \prod_{i=1}^{N} \frac{(2h_i)^{m_i(1)}}{m_i(1)!} E\left[\prod_{j=1}^{N} (B\boldsymbol{\Phi}_j)^{\mathbf{m}'_j}\right] \tag{2.5}$$

where we set $m_i'(1) = m_i(1) + p_i(1)$ and $m_i'(a) = p_i(a)$, $a = 2, 3, 4$.

First, we remark the following fact.[6]

Theorem 2.1.[6] Let $\{\varphi_1, \dots, \varphi_N\}$ be a set of real-valued random variables with joint distribution $d\rho$. Let $\{\varphi_1^{(a)}, \dots, \varphi_N^{(a)}\}$, $a = 1, \dots, 4$, be four independent copies of $\{\varphi_1, \dots, \varphi_N\}$ and define $\boldsymbol{\varphi}_i = (\varphi_i^{(a)}, \dots, \varphi_i^{(a)})$. If $v_1, \dots, v_N \in \mathscr{G}$, then

$$E\left[\prod_{i=1}^{N} (B\boldsymbol{\varphi}_i)^{\mathbf{m}_i}\right] \geqslant 0 \tag{2.6}$$

for all multi-indices $\mathbf{m}_1, \dots, \mathbf{m}_N$.

Theorem 2.1 can be easily derived if we write the expectation $E[F]$ in terms of $E_0[F]$ as

$$E[F] = Z(\{0\})^{-4} E_0\left[F \exp\left(\sum_{1 \leqslant i \leqslant j \leqslant N} J_{ij} \boldsymbol{\varphi}_i \cdot \boldsymbol{\varphi}_j\right)\right] \tag{2.7}$$

and expand the exponential, through (2.3), (2.7), and (2.8).

From Theorem 2.1, we can conclude that for arbitrary sets of four positive integers $\boldsymbol{P} \equiv (P(1), \dots, P(4))$,

$$\int d\mu(\Phi^{(1)}) \cdots d\mu(\Phi^{(4)}) \prod_{a=1}^{4} [(B\boldsymbol{\Phi})^{(a)}]^{P(a)} \geqslant 0 \tag{2.8}$$

if $v_1, \dots, v_N$ belong to the EMN class $\mathscr{G}$. Here $\prod_{a=1}^{4} [(B\boldsymbol{\Phi})^{(a)}]^{P(a)}$ denotes symbolically

$$(B\varphi_{i_1})^{(1)} \cdots (B\varphi_{i_{p(1)}})^{(1)} \times (B\varphi_{j_1})^{(2)} \cdots (B\varphi_{j_{p(2)}})^{(2)}$$
$$\times (B\varphi_{k_1})^{(3)} \cdots (B\varphi_{k_{p(3)}})^{(3)} \times (B\varphi_{l_1})^{(4)} \cdots (B\varphi_{l_{p(4)}})^{(4)} \tag{2.9}$$

Therefore, expanding the product $\prod_{a=1}^{4}[(B\Phi)^{(a)}]^{P(a)}$, we obtain correlation inequalities expressed in terms of the expectation with the measure $d\mu$, corresponding to various choices of $P(1),..., P(4)$.

However, some of them give the trivial inequality, i.e., $0 \geqslant 0$. Hence, to obtain meaningful results we must give the criterion to exclude uninteresting cases. Define the variables $W^{(a)} \equiv (B\Phi)^{(a)}$ and consider the following transformations:

$$\text{(a)}\quad W^{(a)} \to -W^{(a)} \qquad (a=1,..., 4) \tag{2.10a}$$

$$\text{(b)}\quad W^{(2)} \to -W^{(2)}, \qquad W^{(3)} \to -W^{(3)} \tag{2.10b}$$

$$\text{(c)}\quad W^{(2)} \to -W^{(2)}, \qquad W^{(4)} \to -W^{(4)} \tag{2.10c}$$

$$\text{(d)}\quad W^{(3)} \to -W^{(3)}, \qquad W^{(4)} \to -W^{(4)} \tag{2.10d}$$

The transformation (a) corresponds to changing all the signs of $\varphi^{(a)}, a=1,..., 4$ and (b), (c), (d) to exchanging two of four original variables. The measure $d\mu$ is invariant under these transformations. Thus we can conclude that $E[(B\Phi)^{\mathbf{m}}]=0$ unless all m_i are even or all are odd.

In the expansion of $\prod_{a=1}^{4}[(B\Phi)^{(a)}]^{P(a)}$, $4^{P(1)+\cdots+P(4)}$ terms appear. But they can be classified into the set of patterns such that all terms belonging to the same pattern have the same expectation value.

We can make such patterns according to the following procedure.

(1) Choose one term appearing in the expansion of (2.9). Then replace the variable in the first slot by A, whatever it is among $\varphi^{(1)},..., \varphi^{(4)}$. If there are the same sort of variables in other slots, replace all of them by A. If the term contains only one sort of variables, it is changed into $AA\cdots A$ after this step and this procedure completes.

(2) If the term contains more than two sorts of variables, carry out the step (1) again for the remainder of the term, using another character B. After this step, if there still remain unchanged variables, repeat this procedure until all of the variables in the term are replaced by the characters A, B, C, D.

We perform the above procedure by computer programs for all the terms appearing in the expansion of (2.9), because the labor of calculations increases extremely as the number $P(1)+\cdots+P(4)$ increases. Then the terms are classified into patterns expressed by the characters A, B, C, D.

For example, GHS inequality (3.12a) is derived if we choose $\mathbf{P}=(0, 1, 1, 1)$. In this case, the expansion (2.9) reduces to the following form after the above procedures:

$$(1/2^3)(4AAA-4AAB-4ABA-4ABB+8ABC) \tag{2.11}$$

Of $4^3(=64)$ terms appearing in the expansion, the numbers of terms which have the same pattern are 4 for AAA, 12 for AAB, 12 for ABA, 12 for ABB, and 24 for ABC. Smallness of each coefficient is due to the cancellation among the terms which have the same pattern but different signs.

3. NEW CORRELATION INEQUALITIES

We have classified the patterns appearing in the expansion (2.13), according to the procedure given in the previous section. The inequalities for U_5 and U_8 are rather complicated and hence we collect them in the tables. Here the limitations of the range of the possible values of $\mathbf{P}=(P(1),..., P(4))$ is due to the capacity of the computer program.

In the following, we present some of our results together with the possible applications. We define the n-point function

$$\langle \varphi_{i_1} \cdots \varphi_{i_n} \rangle = \frac{\langle \varphi_{i_1} \cdots \varphi_{i_n} \exp(\sum_{i \in A} h_i \varphi_i) \rangle_0}{\langle \exp(\sum_{i \in A} h_i \varphi_i) \rangle_0} \tag{3.1}$$

where the expectation $\langle F \rangle_0$ is defined by

$$\langle F \rangle_0 = \int F(\Phi) \exp\left(\sum_{i,j \in A} J_{ij} \varphi_i \varphi_j \right) \prod_{i \in A} dv_i(\varphi_i) \tag{3.2}$$

For notational simplicity, we use $\langle i_1 \cdots i_n \rangle$ instead of $\langle \varphi_{i_1} \cdots \varphi_{i_n} \rangle$. The connected n-point or Ursell function U_n is defined by

$$U_n(i_1,..., i_n) = \frac{\partial^n}{\partial h_{i_n} \cdots \partial h_{i_n}} \ln \left\langle \exp\left(\sum_{i \in A} h_i \varphi_i \right) \right\rangle_0 \tag{3.3}$$

To be more concrete,

$$U_1(1) = \langle 1 \rangle \tag{3.4}$$

$$U_2(1, 2) = \langle 12 \rangle - \langle 1 \rangle \langle 2 \rangle \tag{3.5}$$

$$U_3(1, 2, 3) = \langle 123 \rangle - \langle 1 \rangle \langle 23 \rangle - \langle 2 \rangle \langle 13 \rangle - \langle 3 \rangle \langle 12 \rangle + 2\langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \tag{3.6}$$

and

$$U_4^{(0)}(1, 2, 3, 4) = \langle 1234 \rangle - \langle 12 \rangle \langle 34 \rangle - \langle 14 \rangle \langle 23 \rangle - \langle 13 \rangle \langle 24 \rangle \tag{3.7}$$

$$U_6^{(0)}(1, 2, 3, 4, 5, 6) = \langle 123456 \rangle - \sum_{\{i,j\} \subset \{1,...,6\}} \langle ij \rangle \langle klmn \rangle + 2 \sum_{\text{pairings}} \langle ij \rangle \langle kl \rangle \langle mn \rangle \tag{3.8}$$

$$U_8^{(0)}(1,..., 8) = \langle 12345678 \rangle - \sum \langle i_1 i_2 \rangle \langle i_3 \cdots i_8 \rangle$$
$$- \sum \langle i_1 \cdots i_4 \rangle \langle i_5 \cdots i_8 \rangle$$
$$+ 2 \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \langle i_5 \cdots i_8 \rangle$$
$$- 6 \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \langle i_5 i_6 \rangle \langle i_7 i_8 \rangle \tag{3.9}$$

where we have used the superscript (0) to indicate the connected n-point function obtained by ommitting the terms which contain at least one expectation $\langle i_1 \cdots i_n \rangle$ with n odd.

Note that

$$U_1(1) \geqslant 0, \qquad U_2(1, 2) \geqslant 0 \tag{3.10a, b}$$

and more generally

$$\langle \varphi^A \rangle \geqslant 0, \qquad \langle \varphi^A; \varphi^B \rangle \equiv \langle \varphi^A \varphi^B \rangle - \langle \varphi^A \rangle \langle \varphi^B \rangle \geqslant 0 \tag{3.11}$$

which are known as the GKS inequalities,[9] respectively.

Now we enumerate new correlation inequalities. [Note that the inequalities (3.12a), (3.13a), and (3.14a) were already derived in Ref. 6, which are presented for completeness.]

Theorem 3.1. If $v_1,..., v_N$ belong to the EMN class, then

$$0 \geqslant U_3(i, j, k) \geqslant -4 \langle i \rangle \langle j; k \rangle \tag{3.12a, b}$$

Theorem 3.2. If $v_1,..., v_N$ belong to the EMN class, then

$$0 \geqslant U_4^{(0)}(i, j, k, l) \geqslant -4 \langle ij \rangle \langle kl \rangle \tag{3.13a, b}$$

Theorem 3.3. If $v_1,..., v_N$ belong to the EMN class, then

$$U_6^{(0)}(i_1,..., i_6) \leqslant -\frac{4}{5} \sum_1 \langle i_1' i_2' \rangle \, U_4^{(0)}(i_3',..., i_6') \tag{3.14a}$$

$$U_6^{(0)}(i_1,..., i_6) \geqslant -\frac{4}{5} \sum_1 \langle i_1' i_2' \rangle \, U_4^{(0)}(i_3',..., i_6')$$
$$- \frac{16}{15} \sum_2 \langle i_1' i_2' \rangle \langle i_3' i_4' \rangle \langle i_5' i_6' \rangle \tag{3.14b}$$

where the sum $\sum_1$ extends over the 15 different partitions of $\{i_1,..., i_6\}$ into subsets $\{i_1', i_2'\}$ and $\{i_3',..., i_6'\}$, and the sum $\sum_2$ over the 15 different partitions of $\{i_1,..., i_6\}$ into subsets $\{i_1', i_2'\}$, $\{i_3', i_4'\}$, and $\{i_5', i_6'\}$.

The obtained inequalities for U_n and corresponding multi-indices $\mathbf{P} = (P(1),..., P(4))$ are collected in Tables I and II. For example, the GHS inequality (3.12a) corresponds to the choice $\mathbf{P} = (0, 1, 1, 1)$ and the Lebowitz inequality (3.13a) to $\mathbf{P} = (1, 1, 1, 1)$. Here we remark that inequalities (3.14a, b) are symmetrized in terms of arguments, using the fact that $\langle ij \rangle$, $U_4(i_1,..., i_4)$ and $U_6(i_1,..., i_6)$ are totally symmetric under the permutations of their arguments. For example, corresponding to $\mathbf{P} = (3, 1, 1, 1)$, we obtain

$$U_6^{(0)}(1,..., 6) \leqslant -4\langle 12 \rangle U_4^{(0)}(3, 4, 5, 6) -4\langle 13 \rangle U_4^{(0)}(2, 4, 5, 6) - 4\langle 23 \rangle U_4^{(0)}(1, 4, 5, 6) \quad (3.15)$$

Table I. Upper and Lower Bounds on the Connected n-Point Function U_n with the Corresponding Multi-Indices $P = (P(1),..., P(4))$.

n	Upper bounds		Lower bounds	
1			(1, 0, 0, 0)	(3.10a)
2			(0, 2, 0, 0) (0, 0, 2, 0) (0, 0, 0, 2)	(3.10b)
3	(0, 1, 1, 1)	(3.12a) GHS	(1, 2, 0, 0) (1, 0, 2, 0) (1, 0, 0, 2)	(3.12b)
4	(1, 1, 1, 1)	(3.13a) Lebowitz	(0, 2, 2, 0) (0, 2, 0, 2) (0, 0, 2, 2) (2, 2, 0, 0) (2, 0, 2, 0) (2, 0, 0, 2)	(3.13b)
			(0, 4, 0, 0) (0, 0, 4, 0) (0, 0, 0, 4)	A1[a]
5	(0, 3, 1, 1) (0, 1, 3, 1) (0, 1, 1, 3)	A2	(1, 0, 2, 2) (1, 2, 0, 2) (1, 2, 2, 0)	A4
	(2, 1, 1, 1)	A3	(1, 4, 0, 0) (1, 0, 4, 0) (1, 0, 0, 4)	A5
			(3, 2, 0, 0) (3, 0, 2, 0) (3, 0, 0, 2)	A6

[a] A1, A2, etc. refer to sections of the Appendix.

Table II. Upper and Lower Bounds on the Connected n-Point Function U_n with the Corresponding Multi-Indices $P=(P(1),\ldots, P(4))$

n	Upper bounds		Lower bounds	
6	(3, 1, 1, 1)	(3.14a)	(0, 2, 2, 2)	(3.14b)
	(1, 3, 1, 1)		(2, 0, 2, 2)	
	(1, 1, 3, 1)		(2, 2, 0, 2)	
	(1, 1, 1, 3)		(2, 2, 2, 0)	
			(0, 2, 4, 0)	A7[a]
			(0, 2, 0, 4)	
			(0, 0, 2, 4)	
			(0, 4, 2, 0)	
			(0, 4, 0, 2)	
			(0, 0, 4, 2)	
			(2, 4, 0, 0)	
			(2, 0, 4, 0)	
			(2, 0, 0, 4)	
			(4, 2, 0, 0)	
			(4, 0, 2, 0)	
			(4, 0, 0, 2)	
			(0, 6, 0, 0)	A8
			(0, 0, 6, 0)	
			(0, 0, 0, 6)	
8	(3, 3, 1, 1)	A9	(2, 2, 2, 2)	A11
	(5, 1, 1, 1)	A10	(4, 2, 2, 0)	A12
			(4, 4, 0, 0)	A13
			(6, 2, 0, 0)	A14

[a] A7, A8, etc. refer to sections of the Appendix.

which reduces to (3.14a) after the above symmetrization. Note that such symmetrizations are performed for the inequality cited in each table.

The inequality (3.13b) leads to the conclusion that the renormalized coupling constant g of $\lambda(\varphi^4)_d$ theory defined by

$$g^{(4)} \equiv |\bar{U}_4|/(\chi^2 \xi^d) \tag{3.16}$$

has the uniform upper bounds, which depends only on the dimension d. The similar result

$$U_4(x_1,\ldots, x_4) \geqslant -2\langle \varphi_{x_1}\varphi_{x_2}\rangle\langle \varphi_{x_3}\varphi_{x_4}\rangle \tag{3.17}$$

was proven by Percus[10] and Aizenman[3] for the classical spin-1/2 Ising model. It can be also proven for the φ^4 field models using the Griffiths–Simon representation. Our derivation of (3.13b) is immediate and (3.13b) holds for more general class in the sense that lattice scalar field models with general potential function $V(\varphi)$ satisfying the condition (c) of Theorem 1.2 of Refs. 6, which cannot necessarily be approximated by the classical Ising model through the Griffiths–Simon representations.

If we assume the inequality[10,11]

$$U_6^{(0)}(1,..., 6) \geqslant 0 \tag{3.18}$$

and combine it with (3.13) and (3.15), we can prove the triviality of the continuum limit of lattice scalar models in $d > 4$ dimensions, which have the potential function of the form

$$V(\varphi^2) = \lambda\varphi^6 + \mu\varphi^4 + \sigma\varphi^2 \tag{3.19}$$

with $\lambda \geqslant 0$, $\mu \geqslant 0$, $\sigma \in R$. Here the triviality implies that the dimensionless four-point coupling constant $g^{(4)}$ and six-point one $g^{(6)} \equiv |\bar{\Gamma}_6|/(\chi^3 \xi^{2d})$ both vanish in the critical limit ($\xi \nearrow \infty$), where Γ_6 is the connected one-particle irreducible part of the six-point function. (See Ref. 12 for details and other applications.)

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APPENDIX

A1. $P = (0, 4, 0, 0), (0, 0, 4, 0), (0, 0, 0, 4)$:

$$U_4^{(0)}(1, 2, 3, 4) + 4 \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \geqslant 0$$

A2. $P = (0, 3, 1, 1), (0, 1, 3, 1), (0, 1, 1, 3)$:

$$U_5(1,..., 5) + \frac{6}{5} \sum U_2(i_1, i_2)\, U_3(i_3, i_4, i_5) \leqslant 0$$

A3. $P = (2, 1, 1, 1)$:

$$U_5(1,..., 5) + \frac{8}{5} \sum \langle i_1 \rangle\, U_4(i_2,..., i_5) + \frac{2}{5} \sum \langle i_1 i_2 \rangle\, U_3(i_3, i_4, i_5)$$
$$+ \frac{6}{5} \sum \langle i_1 \rangle \langle i_2 \rangle\, U_3(i_3, i_4, i_5) \leqslant 0$$

A4. $P=(1,0,2,2),(1,2,0,2),(1,2,2,0)$:

$$U_5(1,\dots,5)+\frac{4}{5}\sum\langle i_1\rangle U_4(i_2,\dots,i_5)+\frac{4}{5}\sum\langle i_1 i_2\rangle U_3(i_3,i_4,i_5)$$
$$-\frac{4}{5}\sum\langle i_1\rangle\langle i_2\rangle U_3(i_3,i_4,i_5)+\frac{16}{15}\sum\langle i_1\rangle\langle i_2 i_3\rangle\langle i_4 i_5\rangle$$
$$-\frac{16}{5}\sum\langle i_1\rangle\langle i_2\rangle\langle i_3\rangle\langle i_4 i_5\rangle+16\langle 1\rangle\langle 2\rangle\langle 3\rangle\langle 4\rangle\langle 5\rangle\geqslant 0$$

A5. $P=(1,4,0,0),(1,0,4,0),(1,0,0,4)$:

$$U_5(1,\dots,5)+\frac{4}{5}\sum\langle i_1\rangle U_4(i_2,\dots,i_5)+\frac{12}{5}\sum\langle i_1 i_2\rangle U_3(i_3,i_4,i_5)$$
$$-\frac{12}{5}\sum\langle i_1\rangle\langle i_2\rangle U_3(i_3,i_4,i_5)+\frac{16}{5}\sum\langle i_1\rangle\langle i_2 i_3\rangle\langle i_4 i_5\rangle$$
$$-\frac{48}{5}\sum\langle i_1\rangle\langle i_2\rangle\langle i_3\rangle\langle i_4 i_5\rangle+48\langle 1\rangle\langle 2\rangle\langle 3\rangle\langle 4\rangle\langle 5\rangle\geqslant 0.$$

A6. $P=(3,2,0,0),(3,0,2,0),(3,0,0,2)$:

$$U_5(1,\dots,5)+\frac{12}{5}\sum\langle i_1\rangle U_4(i_2,\dots,i_5)+\frac{8}{5}\sum\langle i_1 i_2\rangle U_3(i_3,i_4,i_5)$$
$$+\frac{16}{5}\sum\langle i_1\rangle\langle i_2\rangle U_3(i_3,i_4,i_5)+\frac{16}{5}\sum\langle i_1\rangle\langle i_2 i_3\rangle\langle i_4 i_5\rangle$$
$$-\frac{16}{5}\sum\langle i_1\rangle\langle i_2\rangle\langle i_3\rangle\langle i_4 i_5\rangle-16\langle 1\rangle\langle 2\rangle\langle 3\rangle\langle 4\rangle\langle 5\rangle\geqslant 0$$

A7. $P=(0,2,4,0)$, $(0,2,0,4)$, $(0,0,2,4)$, $(0,4,2,0)$, $(0,4,0,2)$, $(0,0,4,2)$, $(2,4,0,0)$, $(2,0,4,0)$, $(2,0,0,4)$, $(4,2,0,0)$, $(4,0,2,0)$, $(4,0,0,2)$:

$$U_6^{(0)}(1,\dots,6)+\frac{28}{15}\sum\langle i_1 i_2\rangle U_4^{(0)}(i_3,\dots,i_6)+\frac{16}{5}\sum\langle i_1 i_2\rangle\langle i_3 i_4\rangle\langle i_5 i_6\rangle\leqslant 0$$

A8. $P=(0,6,0,0),(0,0,6,0),(0,0,0,6)$:

$$U_6^{(0)}(1,\dots,6)+4\sum\langle i_1 i_2\rangle U_4^{(0)}(i_3,\dots,i_6)+16\sum\langle i_1 i_2\rangle\langle i_3 i_4\rangle\langle i_5 i_6\rangle\geqslant 0$$

A9. $P = (3, 3, 1, 1)$:

$$U_8^{(0)}(1,..., 8) + \frac{6}{7} \sum \langle i_1 i_2 \rangle\, U_6^{(0)}(i_3,..., i_8) + \frac{36}{35} \sum U_4^{(0)}(i_1,..., i_4)\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{24}{35} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle\, U_4^{(0)}(i_5,..., i_8) \leqslant 0$$

A10. $P = (5, 1, 1, 1)$:

$$U_8^{(0)}(1,..., 8) + \frac{10}{7} \sum \langle i_1 i_2 \rangle\, U_6^{(0)}(i_3,..., i_8) + \frac{4}{7} \sum U_4^{(0)}(i_1,..., i_4)\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{8}{7} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle\, U_4^{(0)}(i_5,..., i_8) \leqslant 0$$

A11. $P = (2, 2, 2, 2)$:

$$U_8^{(0)}(1,..., 8) + \frac{4}{7} \sum \langle i_1 i_2 \rangle\, U_6^{(0)}(i_3,..., i_8) + \frac{44}{35} \sum U_4^{(0)}(i_1,..., i_4)\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{16}{35} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{64}{105} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \langle i_5 i_6 \rangle \langle i_7 i_8 \rangle \geqslant 0$$

A12. $P = (4, 2, 2, 0)$:

$$U_8^{(0)}(1,..., 8) + \frac{8}{7} \sum \langle i_1 i_2 \rangle\, U_6^{(0)}(i_3,..., i_8) + \frac{28}{35} \sum U_4^{(0)}(i_1,..., i_4)\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{128}{105} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{64}{35} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \langle i_5 i_6 \rangle \langle i_7 i_8 \rangle \geqslant 0.$$

A14. $P = (4, 4, 0, 0)$:

$$U_8^{(0)}(1,..., 8) + \frac{12}{7} \sum \langle i_1 i_2 \rangle\, U_6^{(0)}(i_3,..., i_8) + \frac{76}{35} \sum U_4^{(0)}(i_1,..., i_4)\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{16}{5} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle\, U_4^{(0)}(i_5,..., i_8)$$

$$+ \frac{192}{35} \sum \langle i_1 i_2 \rangle \langle i_3 i_4 \rangle \langle i_5 i_6 \rangle \langle i_7 i_8 \rangle \geqslant 0$$

A14. $P=(6, 2, 0, 0)$:

$$U_8^{(0)}(1,\dots, 8)+\frac{16}{7}\sum \langle i_1 i_2\rangle\, U_6^{(0)}(i_3,\dots, i_8)+\frac{12}{7}\sum U_4^{(0)}(i_1,\dots, i_4)\, U_4^{(0)}(i_5,\dots, i_8)$$

$$+\frac{32}{7}\sum \langle i_1 i_2\rangle\langle i_3 i_4\rangle\, U_4^{(0)}(i_5,\dots, i_8)$$

$$+\frac{64}{7}\sum \langle i_1 i_2\rangle\langle i_3 i_4\rangle\langle i_5 i_6\rangle\langle i_7 i_8\rangle \geqslant 0$$

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